

A NON-LINEAR THEORY OF INFRAHYPERFUNCTIONS

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ABSTRACT. We develop a non-linear theory for infrahyperfunctions (also known as quasianalytic (ultra)distributions [25]). In the hyperfunction case our work can be summarized as follows: We construct a differential algebra that contains the space of hyperfunctions as a linear differential subspace and in which the multiplication of real analytic functions coincides with their ordinary product. Moreover, by proving an analogue of Schwartz's impossibility result for hyperfunctions, we show that this embedding is optimal. Our results fully solve a question raised by M. Oberguggenberger [34, p. 286, Prob. 27.2].

1. INTRODUCTION

The non-linear theory of generalized functions was initiated by J.-F. Colombeau [5, 6] who constructed a differential algebra containing the space of distributions as a linear differential subspace and the space of smooth functions as a subalgebra. The algebras of generalized functions provide a framework for linear equations with strongly singular data and non-linear equations [19, 20]. We refer to the monograph [34] for many interesting applications to the theory of partial differential equations.

On the other hand, for several natural linear problems the space of distributions is not the suitable setting, e.g. Cauchy problems for weakly hyperbolic equations – even with smooth coefficients – are in general not well-posed in the space of distributions [7, 9]. Such considerations motivated the search for and study of new spaces of linear generalized functions like ultradistributions [23] and hyperfunctions [39, 33]. For instance, under suitable conditions the above Cauchy problems become well-posed in spaces of ultradistributions [7, 8, 15], while the space of hyperfunctions is the convenient setting for the treatment of partial differential equations with real analytic coefficients [3, 21].

In [11] we presented a non-linear theory of non-quasianalytic ultradistributions (cf. [13, 18]). The goal of this paper is to further extend these results and develop a non-linear theory for hyperfunctions. In fact, we consider general spaces of infrahyperfunctions of class $\{M_p\}$ [25] for a quasianalytic weight sequence M_p [23] and construct a differential algebra that contains the space of infrahyperfunctions of class $\{M_p\}$ as a differential subspace and in which the multiplication of quasianalytic functions of class

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$\{M_p\}$ coincides with their pointwise product. Moreover, by establishing a Schwartz impossibility type result for infrahyperfunctions, we show that our embeddings are optimal in the sense of being consistent with the pointwise multiplication of ordinary functions. The case $M_p = p!$ corresponds to the hyperfunction case, thereby fully settling a question of M. Oberguggenberger [34, p. 286, Prob. 27.2]. We also believe that our work puts the notion of ‘very weak solution’ introduced by C. Garetto and M. Ruzhansky in their study on well-posedness of weakly hyperbolic equations with time dependent nonregular coefficients [16] in a broader perspective and could possibly lead to a natural framework for extensions of their results.

This paper is organized as follows. In the preliminary Section 2 we collect some useful properties of the spaces of quasianalytic functions and their duals (quasianalytic functionals) that will be used later on in the article. Sheaves of infrahyperfunctions are introduced in Section 3. They were first constructed by L. Hörmander¹ [25] but we give a short proof of their existence based on Hörmander’s support theorem for quasianalytic functionals and a general method for the construction of flabby sheaves due to K. Junker and Y. Ito [27, 28]. In Section 4 we define our algebras of generalized functions, provide a null-characterization for the space of negligible sequences, and show that the totality of our algebras forms a sheaf on \mathbb{R}^d . Due to the absence of non-trivial compactly supported quasianalytic functions the latter is much more difficult to establish than for Colombeau algebras of non-quasianalytic type. The proof is based on some of our recent results on the solvability of the Cousin problem for vector-valued quasianalytic functions [12]. We mention that the motivation for [12] was precisely this problem. In Section 5 we embed the infrahyperfunctions of class $\{M_p\}$ into our algebras and show that the multiplication of ultradifferentiable functions of class $\{M_p\}$ is preserved under the embedding we shall construct. Finally, we present a Schwartz impossibility type result for infrahyperfunctions which shows that our embeddings are optimal.

2. SPACES OF QUASIANALYTIC FUNCTIONS AND THEIR DUALS

In this section we fix the notation and introduce spaces of quasianalytic functions and their duals, the so called spaces of quasianalytic functionals. We also discuss the notion of support for quasianalytic functionals and state a result about the solvability of the Cousin problem for vector-valued quasianalytic functions [12] that will be used in Section 4.

Set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $(M_p)_{p \in \mathbb{N}_0}$ be a weight sequence of positive real numbers and set $m_p := M_p/M_{p-1}$, $p \in \mathbb{N}$. We shall always assume that $M_0 = 1$ and that $m_p \rightarrow \infty$ as $p \rightarrow \infty$. Furthermore, we will make use of various of the following conditions:

- (M.1) $M_p^2 \leq M_{p-1}M_{p+1}$, $p \in \mathbb{N}$,
- (M.2)’ $M_{p+1} \leq AH^{p+1}M_p$, $p \in \mathbb{N}_0$, for some $A, H \geq 1$,
- (M.2) $M_{p+q} \leq AH^{p+q}M_pM_q$, $p, q \in \mathbb{N}_0$, for some $A, H \geq 1$,
- (M.2)* $2m_p \leq Cm_{pQ}$, $p \in \mathbb{N}$, for some $Q \in \mathbb{N}$, $C > 0$.

¹He uses the name quasianalytic distributions in [25]; the terminology infrahyperfunctions comes from [14, 35].

$$(QA) \quad \sum_{p=1}^{\infty} \frac{1}{m_p} = \infty,$$

$$(NE) \quad p! \leq Ch^p M_p, \quad p \in \mathbb{N}_0, \text{ for some } C, h > 0.$$

We write $M_\alpha = M_{|\alpha|}$ for $\alpha \in \mathbb{N}_0^d$. The associated function of M_p is defined as

$$M(t) := \sup_{p \in \mathbb{N}_0} \log \frac{t^p}{M_p}, \quad t > 0,$$

and $M(0) = 0$. We define M on \mathbb{R}^d as the radial function $M(x) = M(|x|)$, $x \in \mathbb{R}^d$. As usual, the relation $M_p \subset N_p$ between two weight sequences means that there are $C, h > 0$ such that $M_p \leq Ch^p N_p$, $p \in \mathbb{N}_0$. The stronger relation $M_p \prec N_p$ means that the latter inequality remains valid for every $h > 0$ and a suitable $C = C_h > 0$. Condition (NE) can therefore be formulated as $p! \subset M_p$. We refer to [23, 2] for the meaning of these conditions and their translation in terms of the associated function. In particular, under $(M.1)$, the assumption $(M.2)$ holds [23, Prop. 3.6] if and only if

$$2M(t) \leq M(Ht) + \log A, \quad t \geq 0.$$

Suppose K is a regular compact subset of \mathbb{R}^d , that is, $\overline{\text{int } K} = K$. For $h > 0$ we write $\mathcal{E}^{M_p, h}(K)$ for the Banach space of all $\varphi \in C^\infty(K)$ such that

$$\|\varphi\|_{K, h} := \sup_{\alpha \in \mathbb{N}_0^d} \sup_{x \in K} \frac{|\varphi^{(\alpha)}(x)|}{h^{|\alpha|} M_\alpha} < \infty.$$

For an open set Ω in \mathbb{R}^d , we define

$$\mathcal{E}^{\{M_p\}}(\Omega) = \varprojlim_{K \Subset \Omega} \varinjlim_{h \rightarrow \infty} \mathcal{E}^{M_p, h}(K).$$

The elements of $\mathcal{E}^{\{M_p\}}(\Omega)$ are called *ultradifferentiable functions of class $\{M_p\}$ (or Roumieu type) in Ω* . When $M_p = p!$ the space $\mathcal{E}^{\{M_p\}}(\Omega)$ coincides with the space $\mathcal{A}(\Omega)$ of real analytic functions in Ω . By the Denjoy-Carleman theorem, (QA) means that there are no non-trivial compactly supported ultradifferentiable functions of class $\{M_p\}$, or equivalently, that a function $\varphi \in \mathcal{E}^{\{M_p\}}(\Omega)$ which vanishes on an open subset Ω' that meets every connected component of Ω , is necessarily identically zero.

It should be mentioned that, under $(M.1)$, $(M.2)$ and (NE) , a weight sequence M_p satisfies $(M.2)^*$ [2, Thm. 14] if and only if $\omega = M$ is a Braun-Meise-Taylor type weight function [1] (as defined in [2, p. 426]). In such a case, we have $\mathcal{E}^{\{M_p\}}(\Omega) = \mathcal{E}_{\{\omega\}}(\Omega)$ as locally convex spaces.

We write \mathcal{R} for the family of all positive real sequences $(r_j)_{j \in \mathbb{N}_0}$ with $r_0 = 1$ that increase (not necessarily strictly) to infinity. This set is partially ordered and directed by the relation $r_j \preceq s_j$, which means that there is an $j_0 \in \mathbb{N}_0$ such that $r_j \leq s_j$ for all $j \geq j_0$. Following Komatsu [24], we removed the non-quasianalyticity assumption in his projective description of $\mathcal{E}^{\{M_p\}}(\Omega)$ and we found in [12] an explicit system of semi-norms generating the topology of $\mathcal{E}^{\{M_p\}}(\Omega)$ also in the quasianalytic case.

Lemma 2.1. ([12, Prop. 4.8]) *Let M_p be a weight function satisfying (M.1), (M.2)', and (NE). A function $\varphi \in C^\infty(\Omega)$ belongs to $\mathcal{E}^{\{M_p\}}(\Omega)$ if and only if*

$$\|\varphi\|_{K,r_j} := \sup_{\alpha \in \mathbb{N}_0^d} \sup_{x \in K} \frac{|\varphi^{(\alpha)}(x)|}{M_\alpha \prod_{j=0}^{|\alpha|} r_j} < \infty,$$

for all $K \Subset \Omega$ and $r_j \in \mathcal{R}$. Moreover, the topology of $\mathcal{E}^{\{M_p\}}(\Omega)$ is generated by the system of semi-norms $\{\|\cdot\|_{K,r_j} : K \Subset \Omega, r_j \in \mathcal{R}\}$.

Suppose that the weight sequence M_p satisfies (M.1), (M.2)', (QA), and (NE). The elements of the dual space $\mathcal{E}'^{\{M_p\}}(\Omega)$ are called *quasianalytic functionals of class $\{M_p\}$* (or *Roumieu type*) in Ω , while those of $\mathcal{A}'(\Omega)$ are called *analytic functionals in Ω* . The properties (M.1), (M.2)', and (NE) imply that the space of entire functions is dense in $\mathcal{E}^{\{M_p\}}(\Omega)$ [25, Prop 3.2]. Hence, for any $\Omega' \subseteq \Omega$ and any other weight sequence N_p with $p! \subset N_p \subset M_p$ we may identify $\mathcal{E}'^{\{M_p\}}(\Omega')$ with a subspace of $\mathcal{E}'^{\{N_p\}}(\Omega)$. In particular, we always have $\mathcal{E}'^{\{M_p\}}(\Omega') \subseteq \mathcal{A}'(\Omega')$.

An *ultradifferential operator of class $\{M_p\}$* is an infinite order differential operator

$$P(D) = \sum_{\alpha \in \mathbb{N}_0^d} a_\alpha D^\alpha, \quad a_\alpha \in \mathbb{C},$$

($D^\alpha = (-i\partial)^\alpha$) where the coefficients satisfy the estimate

$$|a_\alpha| \leq \frac{CL^{|\alpha|}}{M_{|\alpha|}}$$

for every $L > 0$ and some $C = C_L > 0$. If M_p satisfies (M.2), then $P(D)$ acts continuously on $\mathcal{E}^{\{M_p\}}(\Omega)$ and hence it can be defined by duality on $\mathcal{E}'^{\{M_p\}}(\Omega)$.

Next, we discuss the notion of support for quasianalytic functionals. For a compact K in \mathbb{R}^d , we define the space of germs of ultradifferentiable functions on K as

$$\mathcal{E}^{\{M_p\}}[K] = \varinjlim_{K \Subset \Omega} \mathcal{E}^{\{M_p\}}(\Omega),$$

a (DFS)-space. Since

$$\mathcal{E}^{\{M_p\}}(\mathbb{R}^d) \cong \varprojlim_{K \Subset \mathbb{R}^d} \mathcal{E}^{\{M_p\}}[K]$$

as locally convex spaces, and $\mathcal{E}^{\{M_p\}}(\mathbb{R}^d)$ is dense in each $\mathcal{E}^{\{M_p\}}[K]$ we have the following isomorphism of vector spaces

$$\mathcal{E}'^{\{M_p\}}(\mathbb{R}^d) \cong \varinjlim_{K \Subset \mathbb{R}^d} \mathcal{E}'^{\{M_p\}}[K].$$

Let $f \in \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d)$. A compact $K \Subset \mathbb{R}^d$ is said to be a $\{M_p\}$ -carrier of f if $f \in \mathcal{E}'^{\{M_p\}}[K]$. It is well known that for every $f \in \mathcal{A}'(\Omega)$ there is a smallest compact $K \Subset \mathbb{R}^d$ among the $\{p!\}$ -carriers of f , called the *support of f* and denoted by $\text{supp}_{\mathcal{A}'} f$. This essentially follows from the cohomology of the sheaf of germs of analytic functions (see e.g. [33]). An elementary proof based on the properties of the Poisson transform of analytic functionals is given in [26, Sect 9.1]. For a proof based on the heat kernel method see [31]. Hörmander noticed that a similar result holds for

quasianalytic functionals of Roumieu type. More precisely, he showed the following important result:

Proposition 2.2. ([25, Cor. 3.5]) *Let M_p be a weight sequence satisfying (M.1), (M.2)', (QA) and (NE). For every $f \in \mathcal{E}^{\{M_p\}}(\Omega)$ there is a smallest compact set among the $\{M_p\}$ -carriers of f and that set coincides with $\text{supp}_{\mathcal{A}'} f$. We simply denote this set by $\text{supp } f$.*

Finally, we introduce vector-valued quasianalytic functions and state a sufficient condition on the target space for the solvability of the Cousin problem. Let F be a locally convex space. We write $\mathcal{E}^{\{M_p\}}(\Omega; F)$ for the space of all $\varphi \in C^\infty(\Omega; F)$ such that for each continuous semi-norm q on F , $K \Subset \Omega$, and $r_j \in \mathcal{R}$ it holds that

$$q_{K, r_j}(\varphi) := \sup_{\alpha \in \mathbb{N}_0^d} \sup_{x \in K} \frac{q(\varphi^{(\alpha)}(x))}{M_\alpha \prod_{j=0}^{|\alpha|} r_j} < \infty.$$

We endow it with the locally convex topology generated by the system of semi-norms $\{q_{K, r_j} : q \text{ continuous semi-norm on } F, K \Subset \Omega, r_j \in \mathcal{R}\}$.

A Fréchet space E with a generating system of semi-norms $\{\|\cdot\|_k : k \in \mathbb{N}\}$ has the property (DN) [32, p. 368] if

$$(\exists m \in \mathbb{N})(\forall k \in \mathbb{N})(\exists j \in \mathbb{N})(\exists \tau \in (0, 1))(\exists C > 0)$$

$$\|y\|_k \leq C \|y\|_m^{1-\tau} \|y\|_j^\tau, \quad y \in E.$$

We use the notation F'_β for F' endowed with the strong topology.

Proposition 2.3. ([12, Thm. 6.7]) *Let M_p be a weight sequence satisfying (M.1), (M.2)', (QA), and (NE). Let $\Omega \subseteq \mathbb{R}^d$ be open, let $\mathcal{M} = \{\Omega_i : i \in I\}$ be an open covering of Ω , and let F be a (DFS)-space such that F'_β has the property (DN). Suppose $\varphi_{i,j} \in \mathcal{E}^{\{M_p\}}(\Omega_i \cap \Omega_j; F)$, $i, j \in I$, are given F -valued functions such that*

$$\varphi_{i,j} + \varphi_{j,k} + \varphi_{k,i} = 0 \quad \text{on } \Omega_i \cap \Omega_j \cap \Omega_k,$$

for all $i, j, k \in I$. Then, there are $\varphi_i \in \mathcal{E}^{\{M_p\}}(\Omega_i; F)$, $i \in I$, such that

$$\varphi_{i,j} = \varphi_i - \varphi_j \quad \text{on } \Omega_i \cap \Omega_j,$$

for all $i, j \in I$.

Naturally, rephrased in the language of cohomology [33] Proposition 2.3 says that the first cohomology group of the open covering \mathcal{M} with coefficients in the sheaf of F -valued quasianalytic functions $\mathcal{E}^{\{M_p\}}(\cdot; F)$ vanishes, that is, $H^1(\mathcal{M}, \mathcal{E}^{\{M_p\}}(\cdot; F)) = 0$.

3. SHEAVES OF INFRAHYPERFUNCTIONS

In [25] Hörmander constructed a flabby sheaf $\mathcal{B}^{\{M_p\}}$ with the property that its set of sections with support in K , $K \Subset \mathbb{R}^d$, coincides with the space of quasianalytic functionals of class $\{M_p\}$ supported in K . In this section, we give a short proof of Hörmander's result and discuss some of the basic properties of the sheaf $\mathcal{B}^{\{M_p\}}$.

Let X be a topological space and let \mathcal{F} be a sheaf on X . For $U \subseteq X$ open and $A \subseteq U$ we write $\Gamma_A(U, \mathcal{F})$ for the set of sections over U with support in A . Define

$$\Gamma_c(U, \mathcal{F}) = \bigcup_{K \in U} \Gamma_K(U, \mathcal{F}).$$

Proposition 3.1. ([25, Sect. 6]) *Let M_p be a weight sequence satisfying (M.1), (M.2)', (QA), and (NE). Then, there exists an (up to sheaf isomorphism) unique flabby sheaf $\mathcal{B}^{\{M_p\}}$ over \mathbb{R}^d such that*

$$\Gamma_K(\mathbb{R}^d, \mathcal{B}^{\{M_p\}}) = \mathcal{E}'^{\{M_p\}}[K], \quad K \in \mathbb{R}^d.$$

Moreover, for every relatively compact open subset Ω of \mathbb{R}^d , one has

$$\mathcal{B}^{\{M_p\}}(\Omega) = \mathcal{E}'^{\{M_p\}}[\overline{\Omega}] / \mathcal{E}'^{\{M_p\}}[\partial\Omega].$$

We call $\mathcal{B}^{\{M_p\}}$ the *sheaf of infrahyperfunctions of class $\{M_p\}$ (or Roumieu type)*. For $M_p = p!$ this is exactly the sheaf of hyperfunctions \mathcal{B} . Our proof of Proposition 3.1 relies on the following general method for the construction of flabby sheaves with prescribed compactly supported sections, due to Junker and Ito [27, 28] (they used it to construct sheaves of vector-valued (Fourier) hyperfunctions). The idea goes back to Martineau's duality approach to hyperfunctions [30, 40].

Lemma 3.2. ([27, Thm. 1.2]) *Let X be a second countable, locally compact topological space. Assume that for each compact $K \in X$ a Fréchet space F_K is given and that for each two compacts $K_1, K_2 \in X$, $K_1 \subseteq K_2$, there is an injective linear continuous mapping $\iota_{K_2, K_1} : F_{K_1} \rightarrow F_{K_2}$ such that for $K_1 \subseteq K_2 \subseteq K_3$, $\iota_{K_3, K_1} = \iota_{K_3, K_2} \circ \iota_{K_2, K_1}$ and $\iota_{K_1, K_1} = \text{id}$. We shall identify F_{K_1} with its image under the mapping ι_{K_1, K_2} . Suppose that the following conditions are satisfied:*

(FS1) *If $K_1, K_2 \in X$, $K_1 \subseteq K_2$, have the property that every connected component of K_2 meets K_1 , then F_{K_1} is dense in F_{K_2} .*

(FS2) *For $K_1, K_2 \in X$ the mapping*

$$F_{K_1} \times F_{K_2} \rightarrow F_{K_1 \cup K_2} : (f_1, f_2) \rightarrow f_1 + f_2$$

is surjective.

(FS3) (i) *For $K_1, K_2 \in X$ it holds that*

$$F_{K_1 \cap K_2} = F_{K_1} \cap F_{K_2}.$$

(ii) *Let $K_1 \supseteq K_2 \supseteq \dots$ be a decreasing sequence of compacts in X and set $K = \bigcap_n K_n$. Then,*

$$F_K = \bigcap_{n \in \mathbb{N}} F_{K_n}.$$

(FS4) $F_\emptyset = \{0\}$.

Then, there exists an (up to sheaf isomorphism) unique flabby sheaf \mathcal{F} over X such that

$$\Gamma_K(X, \mathcal{F}) = F_K, \quad K \in X.$$

Moreover, for every relatively compact open subset U of X it holds that

$$\mathcal{F}(U) = F_{\overline{U}} / F_{\partial U}.$$

We shall often use the following extension principle.

Lemma 3.3. ([22, Lemma 2.3, p. 226]) *Let X be a second countable topological space and let \mathcal{F} and \mathcal{G} be soft sheaves on X . Let $\rho_c : \Gamma_c(X, \mathcal{F}) \rightarrow \Gamma_c(X, \mathcal{G})$ be a linear mapping such that*

$$\text{supp } \rho_c(T) \subseteq \text{supp } T, \quad T \in \Gamma_c(X, \mathcal{F}).$$

Then, there is a unique sheaf morphism $\rho : \mathcal{F} \rightarrow \mathcal{G}$ such that, for every open set U in X , we have $\rho_U(T) = \rho_c(T)$ for all $T \in \Gamma_c(U, \mathcal{F})$. If, moreover,

$$\text{supp } \rho_c(T) = \text{supp } T, \quad T \in \Gamma_c(X, \mathcal{F}),$$

then ρ is injective.

Proof of Proposition 3.1. The uniqueness of the sheaf $\mathcal{B}^{\{M_p\}}$ follows from Lemma 3.3². For its existence we use Lemma 3.2. Set $X = \mathbb{R}^d$, $F_K = \mathcal{E}'_{\beta}^{\{M_p\}}[K]$, and $\iota_{K_2, K_1} = {}^t r_{K_2, K_1}$ with r_{K_2, K_1} the canonical restriction mapping

$$\mathcal{E}^{\{M_p\}}[K_2] \rightarrow \mathcal{E}^{\{M_p\}}[K_1].$$

Condition (FS1) is a consequence of (QA) while (FS3) and (FS4) are satisfied because of Proposition 2.2. By the well known criterion for surjectivity of continuous linear mappings between Fréchet spaces [42, Thm. 37.2], (FS2) follows from the fact that the mapping

$$\mathcal{E}^{\{M_p\}}[K_1 \cup K_2] \rightarrow \mathcal{E}^{\{M_p\}}[K_1] \times \mathcal{E}^{\{M_p\}}[K_2] : \varphi \rightarrow (r_{K_1 \cup K_2, K_1}(\varphi), r_{K_1 \cup K_2, K_2}(\varphi))$$

is injective and has closed range. \square

Proposition 3.4. *Let M_p be a weight sequence satisfying (M.1), (M.2)', (QA) and (NE).*

- (i) *Let N_p be another weight sequence satisfying (M.1), (M.2)', (QA) and (NE). If $N_p \subset M_p$, then $\mathcal{B}^{\{M_p\}}$ is a subsheaf of $\mathcal{B}^{\{N_p\}}$. In particular, $\mathcal{B}^{\{M_p\}}$ is always a subsheaf of the sheaf of hyperfunctions \mathcal{B} .*
- (ii) *The sheaf of distributions \mathcal{D}' is a subsheaf of $\mathcal{B}^{\{M_p\}}$ and the following diagram commutes*

$$\begin{array}{ccc} \mathcal{D}' & \longrightarrow & \mathcal{B} \\ \downarrow & \nearrow & \\ \mathcal{B}^{\{M_p\}} & & \end{array}$$

- (iii) *For every ultradifferential operator $P(D)$ of class $\{M_p\}$ there is a unique sheaf morphism $P(D) : \mathcal{B}^{\{M_p\}} \rightarrow \mathcal{B}^{\{M_p\}}$ that coincides on $\Gamma_c(\mathbb{R}^d, \mathcal{B}^{\{M_p\}}) = \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d)$ with the usual action of $P(D)$ on quasianalytic functionals of class $\{M_p\}$.*

Proof. In view of Lemma 3.3, (i) follows from Proposition 2.2, (ii) from the fact that the distributional and hyperfunctional support of a distribution coincide, and (iii) holds because

$$\text{supp } P(D)f \subseteq \text{supp } f, \quad f \in \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d).$$

²A flabby sheaf on a paracompact space is soft.

□

4. ALGEBRAS OF GENERALIZED FUNCTIONS OF CLASS $\{M_p\}$

We now introduce differential algebras $\mathcal{G}^{\{M_p\}}(\Omega)$ of generalized functions of class $\{M_p\}$ as quotients of algebras consisting of sequences of ultradifferentiable functions of class $\{M_p\}$ and satisfying an appropriate growth condition. Furthermore, we provide a null characterization of the negligible sequences and discuss the sheaf-theoretic properties of the functor $\Omega \rightarrow \mathcal{G}^{\{M_p\}}(\Omega)$. Unless otherwise stated, throughout this section M_p stands for a weight sequence satisfying (M.1), (M.2), (QA), and (NE).

We define the space of $\{M_p\}$ -moderate sequences as

$$\mathcal{E}_{\mathcal{M}}^{\{M_p\}}(\Omega) = \{(f_n)_n \in \mathcal{E}^{\{M_p\}}(\Omega)^{\mathbb{N}} : (\forall K \Subset \Omega)(\forall \lambda > 0)(\exists h > 0) \sup_{n \in \mathbb{N}} \|f_n\|_{K,h} e^{-M(\lambda n)} < \infty\},$$

and the space of $\{M_p\}$ -negligible sequences as

$$\mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega) = \{(f_n)_n \in \mathcal{E}^{\{M_p\}}(\Omega)^{\mathbb{N}} : (\forall K \Subset \Omega)(\exists \lambda > 0)(\exists h > 0) \sup_{n \in \mathbb{N}} \|f_n\|_{K,h} e^{M(\lambda n)} < \infty\}.$$

Notice that $\mathcal{E}_{\mathcal{M}}^{\{M_p\}}(\Omega)$ is an algebra under pointwise multiplication of sequences, as follows from (M.1) and (M.2), and that $\mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega)$ is an ideal of $\mathcal{E}_{\mathcal{M}}^{\{M_p\}}(\Omega)$. Hence, we can define the algebra $\mathcal{G}^{\{M_p\}}(\Omega)$ of *generalized functions of class $\{M_p\}$* as the factor algebra

$$\mathcal{G}^{\{M_p\}}(\Omega) = \mathcal{E}_{\mathcal{M}}^{\{M_p\}}(\Omega) / \mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega).$$

We denote by $[(f_n)_n]$ the equivalence class of $(f_n)_n \in \mathcal{E}_{\mathcal{M}}^{\{M_p\}}(\Omega)$. Note that $\mathcal{E}^{\{M_p\}}(\Omega)$ can be regarded as a subalgebra of $\mathcal{G}^{\{M_p\}}(\Omega)$ via the constant embedding

$$(4.1) \quad \sigma_{\Omega}(f) := [(f)_n], \quad f \in \mathcal{E}^{\{M_p\}}(\Omega).$$

We also remark that $\mathcal{G}^{\{M_p\}}(\Omega)$ can be endowed with a canonical action of ultradifferential operators $P(D)$ of class $\{M_p\}$. In fact, since $P(D)$ acts continuously on $\mathcal{E}^{\{M_p\}}(\Omega)$, we have that $\mathcal{E}_{\mathcal{M}}^{\{M_p\}}(\Omega)$ and $\mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega)$ are closed under $P(D)$ if we define its action as $P(D)((f_n)_n) := (P(D)f_n)_n$. Consequently, every ultradifferential operator $P(D)$ of class $\{M_p\}$ induces a well defined linear operator

$$P(D) : \mathcal{G}^{\{M_p\}}(\Omega) \rightarrow \mathcal{G}^{\{M_p\}}(\Omega).$$

We now provide a null characterization of the ideal $\mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega)$.

Lemma 4.1. *Let $(f_n)_n \in \mathcal{E}_{\mathcal{M}}^{\{M_p\}}(\Omega)$. Then, $(f_n)_n \in \mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega)$ if and only if for every $K \Subset \Omega$ there is $\lambda > 0$ such that*

$$\sup_{n \in \mathbb{N}} \sup_{x \in K} |f_n(x)| e^{M(\lambda n)} < \infty.$$

Proof. The proof is based on the following multivariable version of Gorny's inequality: Let K and K' be regular compact subsets of \mathbb{R}^d such that $K' \Subset K$. Set $d(K', K^c) = \delta > 0$. Then,

$$(4.2) \quad \max_{|\alpha|=k} \|f^{(\alpha)}\|_{L^\infty(K')} \leq 4e^{2k} \left(\frac{m}{k}\right)^k \|f\|_{L^\infty(K)}^{1-k/m} \times \left(\max \left\{ d^m \max_{|\alpha|=m} \|f^{(\alpha)}\|_{L^\infty(K)}, \frac{\|f\|_{L^\infty(K)} m!}{\delta^m} \right\} \right)^{k/m}$$

for all $f \in C^m(K)$ and $0 < k < m$. We prove (4.2) below, but let us assume it for the moment and show how the result follows from it.

Suppose $(f_n)_n$ satisfies the 0-th order estimate. Let $K' \Subset \Omega$ be an arbitrary regular compact set. Choose a regular compact K such that $K' \Subset K \Subset \Omega$. For every $\lambda_1 > 0$ there are $h_1, C_1 > 0$ such that

$$\|f_n^{(\alpha)}\|_{L^\infty(K)} \leq C_1 h_1^{|\alpha|} M_\alpha e^{M(\lambda_1 n)}, \quad \alpha \in \mathbb{N}_0^d, n \in \mathbb{N},$$

and, there are $\lambda_2, C_2 > 0$ such that

$$\|f_n\|_{L^\infty(K)} \leq C_2 e^{-M(\lambda_2 n)}, \quad n \in \mathbb{N}.$$

Let $\beta \in \mathbb{N}_0^d$, $\beta \neq 0$. Applying (4.2) with $k = |\beta|$ and $m = 2|\beta|$, we obtain

$$\begin{aligned} & \|f_n^{(\beta)}\|_{L^\infty(K')} \\ & \leq 4(2e^2)^{|\beta|} \|f_n\|_{L^\infty(K)}^{1/2} \left(\max \left\{ d^{2|\beta|} \max_{|\alpha|=2|\beta|} \|f_n^{(\alpha)}\|_{L^\infty(K)}, \frac{\|f_n\|_{L^\infty(K)} (2|\beta|)!}{\delta^{2|\beta|}} \right\} \right)^{1/2}. \end{aligned}$$

Combining this with the above inequalities and taking $\lambda_1 := \lambda_2/H$, one finds by (M.2) and (NE) that there are $h, C > 0$ such that

$$\|f_n^{(\beta)}\|_{L^\infty(K')} \leq Ch^{|\beta|} M_\beta e^{-M(\lambda_2 n/H^2)}, \quad \beta \in \mathbb{N}_0^d, n \in \mathbb{N}.$$

Thus $(f_n)_n \in \mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega)$.

We now show (4.2). The one-dimensional Gorny inequality [17, p. 324] states that

$$(4.3) \quad \|g^{(k)}\|_{L^\infty([a,b])} \leq 4e^{2k} \left(\frac{m}{k}\right)^k \|g\|_{L^\infty([a,b])}^{1-k/m} \times \left(\max \left\{ \|g^{(m)}\|_{L^\infty([a,b])}, \frac{\|g\|_{L^\infty([a,b])} 2^m m!}{(b-a)^m} \right\} \right)^{k/m},$$

for all $g \in C^m([a,b])$ and $0 < k < m$. Denote by $\partial/\partial\xi$ the directional derivative in the direction ξ , where $\xi \in \mathbb{R}^d$ is a unit vector. It is shown in [4, Thm. 2.2] that

$$(4.4) \quad \max_{|\alpha|=k} \|f^{(\alpha)}\|_{L^\infty(K')} \leq \sup_{|\xi|=1} \left\| \frac{\partial^k f}{\partial^k \xi} \right\|_{L^\infty(K')}$$

for all $f \in C^k(K')$, $k \in \mathbb{N}$ – it is for this inequality that we need the compact K' to be regular. We write $l(x, \xi)$ for the line in \mathbb{R}^d with direction ξ passing through the point $x \in K'$. Define $g_{x,\xi}(t) = f(x + t\xi)$ for $t \in \{t \in \mathbb{R} : x + t\xi \in K\}$. The latter set always

contains a compact interval $I_{x,\xi} \ni 0$ of length at least 2δ . Inequality (4.3) therefore implies that

$$\begin{aligned} \left\| \frac{\partial^k f}{\partial^k \xi} \right\|_{L^\infty(K')} &= \sup_{x \in K'} |g_{x,\xi}^{(k)}(0)| \leq \sup_{x \in K'} \|g_{x,\xi}^{(k)}\|_{L^\infty(I_{x,\xi})} \\ &\leq \sup_{x \in K'} 4e^{2k} \left(\frac{m}{k}\right)^k \|g_{x,\xi}\|_{L^\infty(I_{x,\xi})}^{1-k/m} \times \\ &\quad \left(\max \left\{ \|g_{x,\xi}^{(m)}\|_{L^\infty(I_{x,\xi})}, \frac{\|g_{x,\xi}\|_{L^\infty(I_{x,\xi})} m!}{\delta^m} \right\} \right)^{k/m} \\ &\leq 4e^{2k} \left(\frac{m}{k}\right)^k \|f\|_{L^\infty(K)}^{1-k/m} \left(\max \left\{ \left\| \frac{\partial^m f}{\partial^m \xi} \right\|_{L^\infty(K)}, \frac{\|f\|_{L^\infty(K)} m!}{\delta^m} \right\} \right)^{k/m}. \end{aligned}$$

The result now follows from (4.4) and the fact that

$$\left\| \frac{\partial^m f}{\partial^m \xi} \right\|_{L^\infty(K)} = \sup_{x \in K} \left| \sum_{j_1=1}^d \cdots \sum_{j_m=1}^d \frac{\partial^m f(x)}{\partial x_{j_1} \cdots \partial x_{j_m}} \xi_{j_1} \cdots \xi_{j_m} \right| \leq d^m \sup_{|\alpha|=m} \|f^{(\alpha)}\|_{L^\infty(K)}.$$

□

Next, we discuss the sheaf properties of $\mathcal{G}^{\{M_p\}}(\Omega)$. Given an open subset Ω' of Ω and $f = [(f_n)_n] \in \mathcal{G}^{\{M_p\}}(\Omega)$, the restriction of f to Ω' is defined as

$$f|_{\Omega'} = [(f_n|_{\Omega'})_n] \in \mathcal{G}^{\{M_p\}}(\Omega').$$

Clearly, the assignment $\Omega \rightarrow \mathcal{G}^{\{M_p\}}(\Omega)$ is a presheaf on \mathbb{R}^d . Our aim in the rest of this section is to show that it is in fact a sheaf. The idea of the proof comes from the theory of hyperfunctions in one dimension: the fact that the hyperfunctions are a sheaf on \mathbb{R} is a direct consequence of the Mittag-Leffler Lemma (= solution of the Cousin problem in one dimension) [33]. Similarly, the solvability of the Cousin problem for the spaces $\mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega)$ would imply that $\mathcal{G}^{\{M_p\}}$ is a sheaf on \mathbb{R}^d . To show the latter, we identify the space $\mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega)$ with a space of vector-valued ultradifferentiable functions of class $\{M_p\}$ with values in an appropriate sequence space and then use Proposition 2.3. We need some preparation: For $\lambda > 0$ we define the following Banach space

$$s^{M_p, \lambda} = \{(c_n)_n \in \mathbb{C}^{\mathbb{N}} : \sigma_\lambda((c_n)_n) := \sup_{n \in \mathbb{N}} |c_n| e^{M(\lambda n)} < \infty\}.$$

Set

$$s^{\{M_p\}} = \varinjlim_{\lambda \rightarrow 0^+} s^{M_p, \lambda},$$

a (DFS) -space. Given a sequence $r_j \in \mathcal{R}$ we denote by M_{r_j} the associated function of the sequence $M_p \prod_{j=0}^p r_j$. In [10, Cor. 3.1] we have shown that a sequence $(c_n)_n \in \mathbb{C}^{\mathbb{N}}$ belongs to $s^{\{M_p\}}$ if and only if

$$\sigma_{r_j}((c_n)_n) := \sup_{n \in \mathbb{N}} |c_n| e^{M_{r_j}(n)} < \infty$$

for all $r_j \in \mathcal{R}$. Moreover, the topology of $s^{\{M_p\}}$ is generated by the system of seminorms $\{\sigma_{r_j} : r_j \in \mathcal{R}\}$. The strong dual of $s^{\{M_p\}}$ given by

$$s'_\beta^{\{M_p\}} = \varprojlim_{\lambda \rightarrow 0^+} s^{M_p, -\lambda},$$

where

$$s^{M_p, -\lambda} = \{(c_n)_n \in \mathbb{C}^\mathbb{N} : \sigma'_\lambda((c_n)_n) := \sup_{n \in \mathbb{N}} |c_n| e^{-M(\lambda n)} < \infty\}, \quad \lambda > 0.$$

Proposition 4.2. *We have*

$$\mathcal{E}_{\mathcal{M}}^{\{M_p\}}(\Omega) = \mathcal{E}^{\{M_p\}}(\Omega; s^{\{M_p\}}) \quad \text{and} \quad \mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega) = \mathcal{E}^{\{M_p\}}(\Omega; s^{\{M_p\}}).$$

The proof Proposition 4.2 is based on the ensuing lemma.

Lemma 4.3. *Let $(a_{p,n})_{p,n \in \mathbb{N}}$ be a double sequence of positive real numbers. Then,*

$$\sup_{p,n \in \mathbb{N}} \frac{a_{p,n} e^{M(\lambda n)}}{h^p} < \infty$$

for some $h, \lambda > 0$ if and only if

$$(4.5) \quad \sup_{p,n \in \mathbb{N}} \frac{a_{p,n} e^{M_{s_j}(n)}}{\prod_{j=0}^p r_j} < \infty$$

for all $r_j, s_j \in \mathcal{R}$.

Proof. The direct implication is clear. Conversely, suppose (4.5) holds for all $r_j, s_j \in \mathcal{R}$. Equivalently,

$$\sup_{p,n,q \in \mathbb{N}} \frac{a_{p,n} n^q}{\prod_{j=0}^p r_j \prod_{j=0}^q s_j M_q} < \infty,$$

for all $r_j, s_j \in \mathcal{R}$. Define

$$b_{p,q} = \frac{1}{M_q} \sup_{n \in \mathbb{N}} a_{p,n} n^q, \quad p, q \in \mathbb{N}.$$

Then,

$$\sup_{p,q \in \mathbb{N}} \frac{b_{p,q}}{\prod_{j=0}^p r_j \prod_{j=0}^q s_j} < \infty,$$

for all $r_j, s_j \in \mathcal{R}$. Hence, [37, Lemma 1] implies that there is $h > 0$ such that

$$\infty > \sup_{p,q \in \mathbb{N}} \frac{b_{p,q}}{h^{p+q}} = \sup_{p,n,q \in \mathbb{N}} \frac{a_{p,n} n^q}{h^{p+q} M_q} = \sup_{p,n \in \mathbb{N}} \frac{a_{p,n} e^{M(n/h)}}{h^p}.$$

□

Proof of Proposition 4.2. We only show the second equality. The first one can be shown by a similar argument (but by using [24, Lemma 3.4] instead of Lemma 4.3). Clearly,

the space $\mathcal{E}^{\{M_p\}}(\Omega; s^{\{M_p\}})$ consists of all $(f_n)_n \in \mathcal{E}^{\{M_p\}}(\Omega)^\mathbb{N}$ such that for all $K \Subset \Omega$, $r_j, s_j \in \mathcal{R}$ it holds that

$$\sup_{\alpha \in \mathbb{N}_0^d} \sup_{x \in K} \frac{\sigma_{s_j}(f_n^{(\alpha)}(x))}{M_\alpha \prod_{j=0}^{|\alpha|} r_j} = \sup_{\alpha \in \mathbb{N}_0^d} \sup_{n \in \mathbb{N}} \frac{\|f_n^{(\alpha)}\|_{L^\infty(K)} e^{M_{s_j}(n)}}{M_\alpha \prod_{j=0}^{|\alpha|} r_j} < \infty.$$

We find that $(f_n)_n \in \mathcal{E}_{\mathcal{M}}^{\{M_p\}}(\Omega)$ by applying Lemma 4.3 to

$$a_{p,n} = \max_{|\alpha| \leq p} \frac{\|f_n^{(\alpha)}\|_{L^\infty(K)}}{M_\alpha}, \quad p, n \in \mathbb{N}.$$

The converse inclusion can be shown similarly. \square

Corollary 4.4. *We have*

$$(4.6) \quad \mathcal{E}_{\mathcal{M}}^{\{M_p\}}(\Omega) = \{(f_n)_n \in \mathcal{E}^{\{M_p\}}(\Omega)^\mathbb{N} : (\forall K \Subset \Omega)(\forall r_j \in \mathcal{R})(\exists s_j \in \mathcal{R}) \\ \sup_{n \in \mathbb{N}} \|f_n\|_{K, r_j} e^{-M_{s_j}(n)} < \infty\},$$

and

$$(4.7) \quad \mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega) = \{(f_n)_n \in \mathcal{E}^{\{M_p\}}(\Omega)^\mathbb{N} : (\forall K \Subset \Omega)(\forall r_j \in \mathcal{R})(\forall s_j \in \mathcal{R}) \\ \sup_{n \in \mathbb{N}} \|f_n\|_{K, r_j} e^{M_{s_j}(n)} < \infty\}.$$

Remark 4.5. Since the structure (choice and order of quantifiers) of the spaces occurring in (4.6) and (4.7) coincides with the structure of the widely accepted definition for spaces of moderate and negligible sequences based on an arbitrary locally convex space [13], Corollary 4.4 may serve to clarify our definition of $\mathcal{G}^{\{M_p\}}(\Omega)$.

In [29, Prop. 4.1] it is shown that for a weight sequence M_p satisfying (M.1) and (M.2)', the space $s_{\beta}^{\{M_p\}}$ satisfies (DN) if and only if (M.2)* holds for M_p . We have set the ground to show that $\mathcal{G}^{\{M_p\}}$ is a sheaf.

Theorem 4.6. *Let M_p be a weight sequence satisfying (M.1), (M.2), (M.2)*, (QA), and (NE). The functor $\Omega \rightarrow \mathcal{G}^{\{M_p\}}(\Omega)$ is a sheaf of differential algebras on \mathbb{R}^d . Furthermore, every ultradifferential operator of class $\{M_p\}$ $P(D) : \mathcal{G}^{\{M_p\}} \rightarrow \mathcal{G}^{\{M_p\}}$ is a sheaf morphism.*

Proof. It is clear that the presheaf $\mathcal{G}^{\{M_p\}}$ satisfies the first sheaf axiom (S1). We now show that it also satisfies the patching condition (S2). Let $\Omega \subseteq \mathbb{R}^d$ be open and let $\{\Omega_i : i \in I\}$ be an open covering of Ω . Suppose $f_i = [(f_{i,n})_n] \in \mathcal{G}^{\{M_p\}}(\Omega_i)$ are given such that $f_i = f_j$ on $\Omega_i \cap \Omega_j$, for all $i, j \in I$. This means that there are $(g_{i,j,n})_n \in \mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega_i \cap \Omega_j)$ such that $g_{i,j,n} = f_{i,n} - f_{j,n}$ on $\Omega_i \cap \Omega_j$, for all $i, j \in I$, $n \in \mathbb{N}$. Since $s^{\{M_p\}}$ is a (DFS)-space and $s_{\beta}^{\{M_p\}}$ has property (DN), Propositions 2.3 and 4.2 imply that there are $(g_{i,n})_n \in \mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega_i)$ such that $g_{i,j,n} = g_{i,n} - g_{j,n}$ on $\Omega_i \cap \Omega_j$, for all $i, j \in I$, $n \in \mathbb{N}$. Hence $f_n(x) = f_{i,n}(x) - g_{i,n}(x)$ if $x \in \Omega_i$, is a well defined function on Ω , for each $n \in \mathbb{N}$, and $(f_n)_n \in \mathcal{E}_{\mathcal{M}}^{\{M_p\}}(\Omega)$. Set $f = [(f_n)_n] \in \mathcal{G}^{\{M_p\}}(\Omega)$. Then, $f|_{\Omega_i} = f_i$

for each $i \in I$. Finally, it is clear from its definition that $P(D) : \mathcal{G}^{\{M_p\}} \rightarrow \mathcal{G}^{\{M_p\}}$ is a sheaf morphism. \square

In the next section we establish a very important property of $\mathcal{G}^{\{M_p\}}$, namely, that it is a soft sheaf.

5. EMBEDDING OF THE SHEAF OF INFRAHYPERFUNCTIONS

In this section we shall embed $\mathcal{B}^{\{M_p\}}(\Omega)$ into $\mathcal{G}^{\{M_p\}}(\Omega)$ in such a way that the multiplication of ultradifferentiable functions of class $\{M_p\}$ is preserved. We proceed in various steps. We first construct a support preserving embedding of $\Gamma_c(\mathbb{R}^d, \mathcal{B}^{\{M_p\}}) = \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d)$ into $\Gamma_c(\mathbb{R}^d, \mathcal{G}^{\{M_p\}})$ by means of convolution with a special analytic mollifier sequence that we construct below. Next, we use Lemma 3.3 together with the sheaf properties of $\mathcal{B}^{\{M_p\}}$ and $\mathcal{G}^{\{M_p\}}$ to extend this embedding to the whole space $\mathcal{B}^{\{M_p\}}$. This requires to prove that $\mathcal{G}^{\{M_p\}}$ is soft. We are very indebted to Hörmander for his “hard analysis” type treatment of quasianalytic functionals. A lot of the techniques used in this section are modifications of the ideas from [25].

We start with a discussion about the mollifier sequences that will be used to embed $\mathcal{E}'^{\{M_p\}}(\mathbb{R}^d)$ into $\Gamma_c(\mathbb{R}^d, \mathcal{G}^{\{M_p\}})$. A sequence $(\chi_n)_{n \in \mathbb{N}}$ in $\mathcal{D}(\Omega)$ is called an *analytic cut-off sequence supported in Ω* [26] if

- (a) $0 \leq \chi_n \leq 1$, $n \in \mathbb{N}$,
- (b) $(\chi_n)_n$ is a bounded sequence in $\mathcal{D}(\Omega)$,
- (c) There is $L \geq 1$ such that

$$\|\chi_n^{(\alpha)}\|_{L^\infty(\mathbb{R}^d)} \leq L(Ln)^{|\alpha|}, \quad n \in \mathbb{N}, \quad |\alpha| \leq n.$$

We call $(\chi_n)_n$ an analytic cut-off sequence for $K \Subset \Omega$ if

- (d) there is an open neighborhood V of K such that $\chi_n \equiv 1$ on V for all $n \in \mathbb{N}$.

It is shown in [26, Thm. 1.4.2] that for every $K \Subset \mathbb{R}^d$ and every open neighborhood Ω of K there is an analytic cut-off sequence for K supported in Ω . The following two simple lemmas are very useful. We fix the constants in the Fourier transform as follows

$$\mathcal{F}(\varphi)(\xi) = \widehat{\varphi}(\xi) = \int_{\mathbb{R}^d} \varphi(x) e^{-ix\xi} dx.$$

Lemma 5.1. (Proof of [25, Thm. 3.4]) *Let M_p be a weight sequence satisfying (M.1) and (NE). Let Ω be a relatively compact open subset in \mathbb{R}^d and let $(\chi_n)_n$ be an analytic cut-off sequence supported in Ω . Let $h > 0$. Then,*

$$|\xi|^n |\widehat{\chi_n \varphi}(\xi)| \leq CL|\Omega| \|\varphi\|_{\overline{\Omega}, h} (\sqrt{d}(h + Lk))^n M_n, \quad n \in \mathbb{N}, \xi \in \mathbb{R}^d,$$

for all $\varphi \in \mathcal{E}^{M_p, h}(\overline{\Omega})$, where $C, k > 0$ are chosen in such a way that

$$p^p \leq Ck^p M_p, \quad p \in \mathbb{N}_0.$$

Lemma 5.2. ([23, Lemma 3.3]) *Let $h > 0$ and let $\varphi \in L^1(\mathbb{R}^d)$. If*

$$\mu_h(\varphi) := \int_{\mathbb{R}^d} |\widehat{\varphi}(\xi)| e^{M(\xi/h)} d\xi < \infty,$$

then $\varphi \in C^\infty(\mathbb{R}^d)$ and

$$\sup_{x \in \mathbb{R}^d} |\varphi^{(\alpha)}(x)| \leq \frac{\mu_h(\varphi) h^{|\alpha|} M_\alpha}{(2\pi)^d}, \quad \alpha \in \mathbb{N}_0^d.$$

Let $(\chi_n)_n$ be an analytic cut-off sequence for $\overline{B}(0, 1)$ consisting of even functions. Define

$$\theta_n(x) = n^d \mathcal{F}^{-1}(\chi_n)(nx) = \mathcal{F}^{-1}(\chi_n(\cdot/n))(x), \quad n \in \mathbb{N}.$$

By Lemma 5.2, $(\theta_n)_n \in \mathcal{E}_{\mathcal{M}}^{\{M_p\}}(\mathbb{R}^d)$. The sequence $(\theta_n)_n$ is called an $\{M_p\}$ -mollifier sequence if, in addition, the following property holds: for every $c > 0$ there are $S, \delta, \gamma > 0$ such that

$$(5.1) \quad \sup_{|x| \geq c} |\theta_n^{(\alpha)}(x)| \leq S e^{-M(\delta n)} \gamma^{|\alpha|} M_\alpha, \quad \alpha \in \mathbb{N}_0^d, n \in \mathbb{N}.$$

The next lemma establishes the existence of such mollifier sequences; in fact, we provide an explicit construction of a $\{p!\}$ -mollifier sequence in its proof.

Lemma 5.3. *For every weight sequence M_p satisfying (M.1) and (NE) there exists an $\{M_p\}$ -mollifier sequence.*

Proof. It suffices to consider the case $M_p = p!$. Moreover, if $(\theta_n)_n$ is a one-dimensional $\{p!\}$ -mollifier sequence, then $(\boldsymbol{\theta}_n)_n$ with

$$\boldsymbol{\theta}_n = \underbrace{\theta_n \otimes \cdots \otimes \theta_n}_{d \text{ times}}$$

is a d -dimensional $\{p!\}$ -mollifier sequence, since $(\theta_n)_n \in \mathcal{E}_{\mathcal{M}}^{\{M_p\}}(\mathbb{R})$. Therefore we may also assume $d = 1$. Let $(\kappa_n)_n$ be an analytic cut-off sequence for $[-2, 2]$ consisting of even functions. We denote by H the characteristic function of $[-1, 1]$. Define

$$H_n = \left(\frac{n}{2}\right)^n \underbrace{H(n \cdot) * \cdots * H(n \cdot)}_{n \text{ times}}, \quad n \in \mathbb{N}.$$

Notice that $\text{supp } H_n \subseteq [-1, 1]$ and $\int_{\mathbb{R}} H_n(x) dx = 1$ for all $n \in \mathbb{N}$. Set $\chi_n = \kappa_n * H_n$. Then, $(\chi_n)_n$ is an analytic cut-off sequence for $[-1, 1]$. We now show that additionally (5.1) is satisfied (for $M_p = p!$ and, thus, $M(t) \asymp t$). Notice that

$$\theta_n(z) = n \mathcal{F}^{-1}(\chi_n)(nz) = n \mathcal{F}^{-1}(\kappa_n)(nz) (\text{sinc}(z))^n, \quad z \in \mathbb{C}, n \in \mathbb{N},$$

where, as usual,

$$\text{sinc}(z) = \frac{\sin z}{z} = \frac{1}{2} \widehat{H}(z), \quad z \in \mathbb{C}.$$

Let $0 < a < \pi$ be arbitrary. Then

$$|\text{sinc}(x)| \leq \text{sinc}(a) =: b < 1, \quad a \leq |x| \leq \pi.$$

Furthermore,

$$|\text{sinc}(x + iy)| \leq \frac{e^{|y|}}{\pi}, \quad |x| \geq \pi.$$

Choose μ such that $\max\{b, 1/\pi\} < \mu < 1$. Then there exists $r > 0$ such that

$$|\text{sinc}(x + iy)| \leq \mu, \quad |x| \geq a, |y| \leq r.$$

Since the sequence $(\kappa_n)_n$ is bounded in $\mathcal{D}(\mathbb{R})$ there are $C, D > 0$ such that

$$|z| |\mathcal{F}^{-1}(\kappa_n)(z)| \leq C e^{D|y|}, \quad z = x + iy \in \mathbb{C}, n \in \mathbb{N}.$$

Choose $0 < r_0 < r$ so small that $e^{Dr_0}\mu < 1$. Combining the above two inequalities, we obtain

$$|\theta_n(x + iy)| \leq \frac{C(e^{Dr_0}\mu)^n}{a}, \quad |x| \geq a, |y| \leq r_0, n \in \mathbb{N}.$$

The Cauchy estimates imply that for every $c > 0$ there are $S, \delta, \gamma > 0$ such that

$$\sup_{|x| \geq c} |\theta_n^{(\alpha)}(x)| \leq S e^{-\delta n} \gamma^\alpha \alpha!, \quad \alpha \in \mathbb{N}_0, n \in \mathbb{N}.$$

□

We are ready to embed $\mathcal{E}'^{\{M_p\}}(\mathbb{R}^d)$ into $\Gamma_c(\mathbb{R}^d, \mathcal{G}^{\{M_p\}})$. Fix an analytic cut-off sequence $(\chi_n)_n$ for $\overline{B}(0, 1)$ consisting of even functions and, say, supported in $B(0, r)$ for some $r > 1$. In addition, suppose that the associated sequence

$$\theta_n(x) = n^d \mathcal{F}^{-1}(\chi_n)(nx), \quad n \in \mathbb{N},$$

is an $\{M_p\}$ -mollifier sequence (we shall freely use the constant r , the constant L in property (c), and the ones appearing in (5.1)). Since θ_n is an entire function, the convolution

$$(f * \theta_n)(x) = \langle f(t), \theta_n(x - t) \rangle, \quad x \in \mathbb{R}^d,$$

is well defined for $f \in \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d)$.

Proposition 5.4. *Let M_p be a weight sequence satisfying (M.1), (M.2), and (NE). Then, the mapping*

$$\iota_c : \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d) \rightarrow \Gamma_c(\mathbb{R}^d, \mathcal{G}^{\{M_p\}}) : f \mapsto \iota_c(f) = [(f * \theta_n)_n],$$

is a linear embedding. Furthermore, $\text{supp } \iota_c(f) = \text{supp } f$ for all $f \in \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d)$.

We need two lemmas in preparation for the proof of Proposition 5.4. As customary, we denote by

$$m(t) := \sum_{m_p \leq t} 1, \quad t \geq 0,$$

the counting function of the sequence $(m_p)_{p \in \mathbb{N}}$. Notice that, by (M.1),

$$\frac{t^{m(t)}}{M_{m(t)}} = \sup_{p \in \mathbb{N}_0} \frac{t^p}{M_p} = e^{M(t)}, \quad t \geq 0.$$

Lemma 5.5. *Let $\Omega \subseteq \mathbb{R}^d$ be open and let Ω' be a relatively compact open subset of Ω . Let $\kappa \in \mathcal{D}(\Omega)$ such that $0 \leq \kappa \leq 1$ and $\kappa \equiv 1$ on a neighborhood of $\overline{\Omega'}$. Then,*

$$((\kappa\varphi) * \theta_n - \varphi)_n \in \mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega'),$$

for all $\varphi \in \mathcal{E}^{\{M_p\}}(\Omega)$. In particular,

$$(\kappa\varphi) * \theta_n \rightarrow \varphi, \quad \text{as } n \rightarrow \infty, \text{ in } \mathcal{E}^{\{M_p\}}(\Omega'),$$

Proof. Fix $\varphi \in \mathcal{E}^{\{M_p\}}(\Omega)$. We claim that

$$(5.2) \quad ((\kappa\varphi) * \theta_n - (\kappa_n\varphi) * \theta_n)_n \in \mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega')$$

for any bounded sequence $(\kappa_n)_n$ in $\mathcal{D}(\Omega)$ such that $0 \leq \kappa_n \leq 1$ and for which there is a neighborhood U of $\overline{\Omega'}$ such that $\kappa_n \equiv 1$ on U for all $n \in \mathbb{N}$. Before proving the claim, we show how it implies the result. Let $(\rho_p)_p$ be an analytic cut-off sequence for $\overline{\Omega'}$ supported in Ω . Lemma 5.1 implies that there are $C, h > 0$ such that

$$|\xi|^p |\widehat{\rho_p \varphi}(\xi)| \leq Ch^p M_p, \quad p \in \mathbb{N}, \xi \in \mathbb{R}^d.$$

Set $p_n = m(n/(Hh)) + d + 1$ and $\kappa_n = \rho_{p_n}$, $n \in \mathbb{N}$. By the claim it suffices to show that

$$((\kappa_n\varphi) * \theta_n - \varphi)_n \in \mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega').$$

For $K \Subset \Omega'$ we have

$$\begin{aligned} \sup_{x \in K} |((\kappa_n\varphi) * \theta_n)(x) - \varphi(x)| &\leq \sup_{x \in K} |((\kappa_n\varphi) * \theta_n)(x) - (\kappa_n\varphi)(x)| \\ &\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\widehat{\kappa_n \varphi}(\xi)| (1 - \chi_n(\xi/n)) d\xi \\ &\leq \frac{AC(Hh)^{d+1} M_{d+1}}{(2\pi)^d} \int_{|\xi| \geq n} \frac{(Hh)^{m(n/(Hh))} M_{m(n/(Hh))}}{|\xi|^{m(n/(Hh))+d+1}} d\xi \\ &\leq De^{-M(n/(Hh))}, \end{aligned}$$

where

$$D = \frac{AC(Hh)^{d+1} M_{d+1}}{(2\pi)^d} \int_{|\xi| \geq 1} \frac{1}{|\xi|^{d+1}} d\xi < \infty,$$

whence the result follows (cf. Lemma 4.1). We now show (5.2). We first prove that

$$((\kappa_n\varphi) * \theta_n)_n \in \mathcal{E}_{\mathcal{M}}^{\{M_p\}}(\mathbb{R}^d).$$

Since the sequence $(\kappa_n\varphi)_n$ is bounded in $\mathcal{D}(\mathbb{R}^d)$ there is $C > 0$ such that

$$|\widehat{\kappa_n \varphi}(\xi)| \leq \frac{C}{(1 + |\xi|)^{d+1}}, \quad \xi \in \mathbb{R}^d, n \in \mathbb{N}.$$

For $\lambda > 0$ we have

$$\mu_{1/\lambda}((\kappa_n\varphi) * \theta_n) = \int_{\mathbb{R}^d} |\widehat{\kappa_n \varphi}(\xi)| \chi_n(\xi/n) e^{M(\lambda\xi)} d\xi \leq Ce^{M(\lambda rn)} \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi|)^{d+1}} d\xi,$$

and therefore the sequence is moderate by Lemma 5.2. Hence, by Lemma 4.1, it suffices to show that for every $K \Subset \Omega'$ there is $\lambda > 0$ such that

$$\sup_{n \in \mathbb{N}} \sup_{x \in K} |((\kappa\varphi) * \theta_n)(x) - ((\kappa_n\varphi) * \theta_n)(x)| e^{M(\lambda n)} < \infty.$$

Choose $c > 0$ such that $\kappa - \kappa_n \equiv 0$ on $K + B(0, c)$ and $K_0 \Subset \Omega$ such that $\text{supp}(\kappa - \kappa_n) \subseteq K_0$ for all $n \in \mathbb{N}$. Then,

$$\begin{aligned} \sup_{x \in K} |((\kappa\varphi) * \theta_n)(x) - ((\kappa_n\varphi) * \theta_n)(x)| &\leq \sup_{x \in K} \left| \int_{\mathbb{R}^d} (\kappa(x-t) - \kappa_n(x-t)) \varphi(x-t) \theta_n(t) dt \right| \\ &\leq 2S \|\varphi\|_{L^\infty(K_0)} |K_0| e^{-M(\delta n)}. \end{aligned}$$

□

For $h > 0$, we write $\mathcal{E}_\infty^{M_p, h}(\mathbb{R}^d)$ for the Banach space consisting of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\sup_{\alpha \in \mathbb{N}_0^d} \sup_{x \in \mathbb{R}^d} \frac{|\varphi^{(\alpha)}(x)|}{h^{|\alpha|} M_\alpha} < \infty.$$

Lemma 5.6. (cf. first part of [25, Thm. 3.4]) *Let $h > 0$ and let Ω be a relatively compact open subset of \mathbb{R}^d . Suppose that $(\kappa_p)_p$ is an analytic cut-off sequence supported in Ω and that $(\psi_n)_n$ is a uniformly bounded sequence of continuous functions on \mathbb{R}^d such that $\text{supp}(\psi_1) \subseteq B(0, r)$ and*

$$\text{supp}(\psi_n) \subseteq B(0, rn) \setminus \overline{B}(0, n-1), \quad n \geq 2.$$

Then, there are a sequence $(p_n)_n$ of natural numbers and $k > 0$ such that

$$R(\varphi) := \sum_{n=1}^{\infty} (\kappa_{p_n} \varphi) * \mathcal{F}^{-1}(\psi_n) \in \mathcal{E}_\infty^{M_p, k}(\mathbb{R}^d), \quad \varphi \in \mathcal{E}^{M_p, h}(\overline{\Omega}).$$

Moreover, the convergence of the series $R(\varphi)$ holds in the topology of $\mathcal{E}_\infty^{M_p, k}(\mathbb{R}^d)$ and the mapping $R : \mathcal{E}^{M_p, h}(\overline{\Omega}) \rightarrow \mathcal{E}_\infty^{M_p, k}(\mathbb{R}^d)$ is continuous.

Proof. By Lemma 5.1 there are $C, k_0 > 0$

$$|\xi|^p |\widehat{\kappa_p \varphi}(\xi)| \leq C \|\varphi\|_{\overline{\Omega}, h} k_0^p M_p, \quad p \in \mathbb{N}, \xi \in \mathbb{R}^d,$$

for all $\varphi \in \mathcal{E}^{M_p, h}(\overline{\Omega})$. For $p = m(t)$, $t \geq 0$, and $k_0 t \leq |\xi| \leq 2rk_0 t$, we obtain

$$|\widehat{\kappa_{m(t)} \varphi}(\xi)| \leq C \|\varphi\|_{\overline{\Omega}, h} e^{-M(|\xi|/(2k_0 r))}.$$

Set $p_n = \max\{1, m((n-1)/k_0)\}$, $n \in \mathbb{N}$. Hence

$$|\widehat{\kappa_{p_n} \varphi}(\xi)| \leq C \|\varphi\|_{\overline{\Omega}, h} e^{-M(\xi/(2k_0 r))}, \quad (n-1) \leq |\xi| \leq rn, n \geq k_0 + 1.$$

Choose $C' > 0$ such that $\|\psi_n\|_{L^\infty(\mathbb{R}^d)} \leq C'$ for all $n \in \mathbb{N}$. Then, for $k = 2H^2 k_0 r$, we have

$$\begin{aligned} \mu_k \left(\sum_{n \geq k_0+1}^{\infty} (\kappa_{p_n} \varphi) * \mathcal{F}^{-1}(\psi_n) \right) &\leq \sum_{n \geq k_0+1}^{\infty} \int_{\mathbb{R}^d} |\psi_n(\xi)| |\widehat{\kappa_{p_n} \varphi}(\xi)| e^{M(\xi/k)} d\xi \\ &\leq C' C \|\varphi\|_{\overline{\Omega}, h} \sum_{n \geq k_0+1}^{\infty} \int_{(n-1) \leq |\xi| \leq rn} e^{M(\xi/k) - M(\xi/(2k_0 r))} d\xi \\ &\leq D \|\varphi\|_{\overline{\Omega}, h}, \end{aligned}$$

where

$$D = A^2 C' C \sum_{n=k_0}^{\infty} e^{-M(n/k)} \int_{\mathbb{R}^d} e^{-M(\xi/(2Hk_0 r))} d\xi < \infty.$$

We also have

$$\mu_k((\kappa_{p_n} \varphi) * \mathcal{F}^{-1}(\psi_n)) \leq C' e^{M(rk_0/k)} |B(0, rk_0)| \|\varphi\|_{L^\infty(\Omega)},$$

for $n \leq k_0$. The result now follows from Lemma 5.2. □

Proof of Proposition 5.4. STEP I: $(f * \theta_n)_n \in \mathcal{E}_{\mathcal{M}}^{\{M_p\}}(\mathbb{R}^d)$ for all $f \in \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d)$: Let $\lambda > 0$ be arbitrary. Choose $C > 0$ such that

$$|\widehat{f}(\xi)| \leq C e^{M(\lambda\xi/H)}, \quad \xi \in \mathbb{R}^d.$$

We have

$$\mu_{1/\lambda}(f * \theta_n) = \int_{\mathbb{R}^d} |\widehat{f}(\xi)| \chi_n(\xi/n) e^{M(\lambda\xi)} d\xi \leq C \int_{|\xi| \leq rn} e^{M(\lambda\xi/H) + M(\lambda\xi)} d\xi \leq D e^{M(H\lambda rn)},$$

where

$$D = A^2 C \int_{\mathbb{R}^d} e^{-M(\lambda\xi/H)} d\xi < \infty.$$

The result follows from Lemma 5.2.

STEP II: $\text{supp } \iota_c(f) \subseteq \text{supp } f$ for all $f \in \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d)$: Let $K \Subset \mathbb{R}^d \setminus \text{supp } f$ be arbitrary. Choose $K' \Subset \mathbb{R}^d$ such that a neighborhood of $\text{supp } f$ is contained in K' and $K' \cap K = \emptyset$. Set $c := d(K, K') > 0$. The continuity of f implies that for each $h > 0$ there is $C > 0$ such that

$$\sup_{x \in K} |(f * \theta_n)(x)| \leq C \|\theta_n\|_{K-K', h}.$$

By Property (5.1) we have that

$$\|\theta_n\|_{K-K', \gamma} = \sup_{\alpha \in \mathbb{N}_0^d} \sup_{x \in K-K'} \frac{|\theta_n^{(\alpha)}(x)|}{\gamma^{|\alpha|} M_\alpha} \leq S e^{-M(\delta n)}.$$

We then obtain that $\iota_c(f)$ vanishes on $\mathbb{R}^d \setminus \text{supp } f$ from Lemma 4.1.

STEP III: $\text{supp } f \subseteq \text{supp } \iota_c(f)$ for all $f \in \mathcal{E}'^{\{M_p\}}(\mathbb{R}^d)$: Set $K = \text{supp } \iota_c(f)$. We need to show that $f \in \mathcal{E}'^{\{M_p\}}(\Omega)$ for every open set $\Omega \ni K$. Fix such a set Ω . By Lemma 2.1 and [38, Lemma 2.3] there is a weight sequence N_p satisfying (M.1), (M.2), and $M_p \prec N_p$ such that $f \in \mathcal{E}'^{\{N_p\}}(\mathbb{R}^d)$. Choose $\rho \in \mathcal{D}(\mathbb{R}^d)$ such that $0 \leq \rho \leq 1$ and $\rho \equiv 1$ on a neighborhood of $\text{supp } f$. Employing Lemma 5.5, we deduce that

$$\langle f, \varphi \rangle = \lim_{n \rightarrow \infty} \langle f, (\rho\varphi) * \theta_n \rangle = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} (f * \theta_n)(x) \rho(x) \varphi(x) dx, \quad \varphi \in \mathcal{E}^{\{M_p\}}(\mathbb{R}^d).$$

Next, choose $\kappa \in \mathcal{D}(\Omega)$ such that $0 \leq \kappa \leq 1$ and $\kappa \equiv 1$ on a neighborhood of K . By STEP II we have that $\kappa - \rho \equiv 0$ on a neighborhood of K . Therefore, $\iota_c(f)|_{\mathbb{R}^d \setminus K} = 0$ implies that $f * \theta_n \rightarrow 0$, as $n \rightarrow \infty$, uniformly on $\text{supp}(\kappa - \rho)$. Hence,

$$(5.3) \quad \langle f, \varphi \rangle = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} (f * \theta_n)(x) \kappa(x) \varphi(x) dx, \quad \varphi \in \mathcal{E}^{\{M_p\}}(\mathbb{R}^d).$$

We now invoke Lemma 5.6. Set $\psi_1 = \chi_1$ and

$$\psi_n = \chi_n \left(\frac{\cdot}{n} \right) - \chi_{n-1} \left(\frac{\cdot}{n-1} \right), \quad n \geq 2.$$

Clearly, the sequence $(\psi_n)_n$ satisfies the requirements of Lemma 5.6. Choose a relatively compact open subset $\Omega' \Subset \Omega$ and an analytic cut-off sequence $(\kappa_p)_p$

for K supported in Ω' . According to Lemma 5.6 (applied to the weight sequence N_p), there are a sequence $(p_n)_n$ and $k > 0$ such that the mapping

$$R : \mathcal{E}^{N_p,1}(\overline{\Omega'}) \rightarrow \mathcal{E}_{\infty}^{N_p,k}(\mathbb{R}^d) : \varphi \rightarrow R(\varphi) = \sum_{n=1}^{\infty} (\kappa_{p_n} \varphi) * \mathcal{F}^{-1}(\psi_n)$$

is continuous. Consider the following continuous inclusion mappings

$$\iota_1 : \mathcal{E}^{\{M_p\}}(\Omega) \rightarrow \mathcal{E}^{N_p,1}(\overline{\Omega'}), \quad \iota_2 : \mathcal{E}_{\infty}^{N_p,k}(\mathbb{R}^d) \rightarrow \mathcal{E}^{\{N_p\}}(\mathbb{R}^d),$$

and set $T = \iota_2 \circ R \circ \iota_1 : \mathcal{E}^{\{M_p\}}(\Omega) \rightarrow \mathcal{E}^{\{N_p\}}(\mathbb{R}^d)$. Equality (5.3) gives us

$$\begin{aligned} \langle f, \varphi \rangle &= \int_{\mathbb{R}^d} \sum_{n=1}^{\infty} (f * \mathcal{F}^{-1}(\psi_n))(x) (\kappa(x) - \kappa_{p_n}(x)) \varphi(x) dx + \\ &\quad \sum_{n=1}^{\infty} \int_{\mathbb{R}^d} (f * \mathcal{F}^{-1}(\psi_n))(x) \kappa_{p_n}(x) \varphi(x) dx, \end{aligned}$$

for all $\varphi \in \mathcal{E}^{\{M_p\}}(\mathbb{R}^d)$. Since $(\kappa - \kappa_{p_n})_n$ is a bounded sequence in $\mathcal{D}(\Omega \setminus K)$, the assumption $\iota_c(f)|_{\mathbb{R}^d \setminus K} = 0$ yields

$$g = \sum_{n=1}^{\infty} (f * \mathcal{F}^{-1}(\psi_n))(\kappa - \kappa_{p_n}) \in \mathcal{D}(\Omega).$$

For the second term, we have

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}^d} (f * \mathcal{F}^{-1}(\psi_n))(x) \kappa_{p_n}(x) \varphi(x) dx = \sum_{n=1}^{\infty} \langle f, \mathcal{F}^{-1}(\psi_n) * (\kappa_{p_n} \varphi) \rangle = \langle f, T(\varphi) \rangle,$$

for all $\varphi \in \mathcal{E}^{\{M_p\}}(\mathbb{R}^d)$. Hence $f = g + {}^tT(f) \in \mathcal{E}'^{\{M_p\}}(\Omega)$. \square

Proposition 5.7. *Let M_p be a weight sequence satisfying (M.1), (M.2), (M.2)*, (QA), and (NE). Then, the sheaf $\mathcal{G}^{\{M_p\}}$ is soft.*

Proof. Let $K \Subset \mathbb{R}^d$ and $K \Subset \Omega$, Ω open, be arbitrary. Choose a relatively compact open set Ω' such that $K \Subset \Omega' \Subset \Omega$. It suffices to show that for every $f = [(f_n)_n] \in \mathcal{G}^{\{M_p\}}(\Omega)$ there is $g = [(g_n)_n] \in \mathcal{G}^{\{M_p\}}(\mathbb{R}^d)$ such that $g = f$ on Ω' . Let $(\kappa_p)_p$ be an analytic cut-off sequence for $\overline{\Omega'}$ supported in Ω . Lemma 5.1 and $(f_n)_n \in \mathcal{E}_{\mathcal{M}}^{\{M_p\}}(\Omega)$ imply that there are $C, h > 0$ such that

$$|\xi|^p |\widehat{\kappa_p f_n}(\xi)| \leq C e^{M(n)} h^p M_p, \quad \xi \in \mathbb{R}^d, p, n \in \mathbb{N}.$$

Let $\psi \in \mathcal{D}(B(0, 2))$ such that $0 \leq \psi \leq 1$ and $\psi \equiv 1$ on $\overline{B}(0, 1)$. Set $a = H^2 h$ and

$$\psi_n = \psi \left(\frac{\cdot}{an} \right), \quad n \in \mathbb{N},$$

and define

$$g_n = (\kappa_{p_n} f_n) * \mathcal{F}^{-1}(\psi_n), \quad n \in \mathbb{N},$$

where $p_n = m(Hn) + d + 1$. We first show that $(g_n)_n \in \mathcal{E}_{\mathcal{M}}^{\{M_p\}}(\mathbb{R}^d)$. Since the sequence $(\kappa_p)_p$ is bounded in $\mathcal{D}(\Omega)$ and $(f_n)_n \in \mathcal{E}_{\mathcal{M}}^{\{M_p\}}(\Omega)$, we have

$$|\widehat{\kappa_p f_n}(\xi)| \leq \frac{D_\lambda e^{M(\lambda n)}}{(1 + |\xi|)^{d+1}}, \quad \xi \in \mathbb{R}^d, p, n \in \mathbb{N},$$

for every $\lambda > 0$ and suitable $D_\lambda > 0$. Hence

$$\mu_{1/\lambda}(g_n) = \int_{\mathbb{R}^d} |\widehat{\kappa_{p_n} f_n}(\xi)| \psi_n(\xi) e^{M(\lambda \xi)} d\xi \leq AD_{2a\lambda} e^{M(2Ha\lambda n)} \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi|)^{d+1}} d\xi$$

for all $\lambda > 0$. The sequence $(g_n)_n$ is therefore moderate by Lemma 5.2. We still need to show that

$$(g_n - f_n)_n \in \mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega').$$

We verify this via Lemma 4.1. For any $K \Subset \Omega'$, we have

$$\begin{aligned} \sup_{x \in K} |g_n(x) - f_n(x)| &\leq \sup_{x \in K} |((\kappa_{p_n} f_n) * \mathcal{F}^{-1}(\psi_n))(x) - (\kappa_{p_n} f_n)(x)| \\ &\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\widehat{\kappa_{p_n} f_n}(\xi)| (1 - \psi_n(\xi)) d\xi \\ &\leq \frac{AC(Hh)^{d+1} M_{d+1} e^{M(n)}}{(2\pi)^d} \int_{|\xi| \geq an} \frac{(Hh)^{m(Hn)} M_{m(Hn)}}{|\xi|^{m(Hn)+d+1}} d\xi \\ &\leq D e^{-M(n)}, \end{aligned}$$

where

$$D = \frac{A^2 C (Hh)^{d+1} M_{d+1}}{(2\pi)^d} \int_{|\xi| \geq a} \frac{1}{|\xi|^{d+1}} d\xi < \infty,$$

whence the result follows. \square

We have completed all necessary work to prove our main theorem. Recall that $\sigma_\Omega : \mathcal{E}^{\{M_p\}}(\Omega) \rightarrow \mathcal{G}^{\{M_p\}}(\Omega)$ stands for the constant embedding (4.1).

Theorem 5.8. *Let M_p be a weight sequence satisfying (M.1), (M.2), (M.2)*, (QA), and (NE). Then, there is a unique injective sheaf morphism $\iota : \mathcal{B}^{\{M_p\}} \rightarrow \mathcal{G}^{\{M_p\}}$ such that for each open set $\Omega \subseteq \mathbb{R}^d$ the following properties hold:*

- (i) $\iota_\Omega(f) = \iota_c(f)$ for all $f \in \mathcal{E}^{\{M_p\}}(\Omega)$,
- (ii) for all ultradifferential operators $P(D)$ of class $\{M_p\}$ we have

$$P(D)\iota_\Omega(f) = \iota_\Omega(P(D)f), \quad f \in \mathcal{B}^{\{M_p\}}(\Omega).$$

- (iii) $\iota_\Omega(f) = \sigma_\Omega(f)$ for all $f \in \mathcal{E}^{\{M_p\}}(\Omega)$. Consequently,

$$\iota_\Omega(fg) = \iota_\Omega(f)\iota_\Omega(g), \quad f, g \in \mathcal{E}^{\{M_p\}}(\Omega).$$

Proof. Since $\mathcal{B}^{\{M_p\}}$ and $\mathcal{G}^{\{M_p\}}$ are soft sheaves (Propositions 3.1 and 5.7), the existence and uniqueness of a sheaf embedding ι satisfying properties (i) and (ii) is clear from Lemma 3.3 and Proposition 5.4. We now show that property (iii) is also satisfied. It suffices to show that $\iota_\Omega(f)|_{\Omega'} = \sigma_\Omega(f)|_{\Omega'}$ for all open sets $\Omega' \Subset \Omega$. Fix such a set Ω' and choose $\kappa \in \mathcal{D}(\Omega)$ such that $0 \leq \kappa \leq 1$ and $\kappa \equiv 1$ on a neighborhood of $\overline{\Omega'}$. Notice that

$\iota_\Omega(f)|_{\Omega'} = \iota_{\Omega'}(f|_{\Omega'}) = \iota_{\Omega'}((\kappa f)|_{\Omega'}) = \iota_\Omega(\kappa f)|_{\Omega'} = \iota_c(\kappa f)|_{\Omega'}$. Hence it suffices to show that

$$((\kappa f) * \theta_n - f)_n \in \mathcal{E}_{\mathcal{N}}^{\{M_p\}}(\Omega'),$$

but this has already been proved in Lemma 5.5. \square

We end this article with a remark concerning the optimality of the above embedding. It can be viewed as an analogue of Schwartz's impossibility result [41] in the setting of infrahyperfuctions, which states that, under some natural assumptions, the property (iii) from Theorem 5.8 *cannot* be improved to also preserve the multiplication of all ultradifferentiable functions from a class with lower regularity than $\mathcal{E}^{\{M_p\}}(\Omega)$.

Remark 5.9. Let N_p be another weight sequence satisfying (M.1) and (M.2)'. When embedding $\mathcal{B}^{\{M_p\}}(\Omega)$ into an associative and commutative algebra $(\mathcal{A}^{\{M_p\}, \{N_p\}}(\Omega), +, \circ)$, the following requirements seem to be natural:

- (P.1) $\mathcal{B}^{\{M_p\}}(\Omega)$ is linearly embedded into $\mathcal{A}^{\{M_p\}, \{N_p\}}(\Omega)$ and $f(x) \equiv 1$ is the unity in $\mathcal{A}^{\{M_p\}, \{N_p\}}(\Omega)$.
- (P.2) For each ultradifferential operator $P(D)$ of class $\{M_p\}$ there is a linear operator $P(D) : \mathcal{A}^{\{M_p\}, \{N_p\}}(\Omega) \rightarrow \mathcal{A}^{\{M_p\}, \{N_p\}}(\Omega)$ satisfying the following generalized Leibniz' rule

$$P(D)(q \circ f) = \sum_{\beta \leq \deg q} \frac{1}{\beta!} D^\beta q \circ (D^\beta P)(D)f,$$

for every $f \in \mathcal{A}^{\{M_p\}, \{N_p\}}(\Omega)$ and every polynomial q . Moreover, $P(D)|_{\mathcal{B}^{\{M_p\}}(\Omega)}$ coincides with the usual action of $P(D)$ on infrahyperfuctions of class $\{M_p\}$.

- (P.3) $\circ|_{\mathcal{E}^{\{N_p\}}(\Omega) \times \mathcal{E}^{\{N_p\}}(\Omega)}$ coincides with the pointwise product of functions.

The ensuing result imposes a limitation on the possibility of constructing such an algebra if $M_p \prec N_p$ (implying that $\mathcal{E}^{\{M_p\}}(\Omega) \subsetneq \mathcal{E}^{\{N_p\}}(\Omega)$). On the other hand, our differential algebra $\mathcal{G}^{\{M_p\}}(\Omega)$ satisfies all the properties (P.1)-(P.3) for $M_p = N_p$ and therefore Theorem 5.8 is optimal in this sense.

Theorem 5.10. *Let $M_p \prec N_p$. Then, there is no associative and commutative algebra $\mathcal{A}^{\{M_p\}, \{N_p\}}(\Omega)$ satisfying (P.1)-(P.3).*

Proof. The proof is basically the same as that of the corresponding result for non-quasianalytic ultradistributions [11, Thm. 3.1], but we include it for the sake of completeness. Suppose $\mathcal{A}^{\{M_p\}, \{N_p\}}(\Omega)$ is such an algebra. We have that $q \circ P(D)g = qP(D)g$ for every ultradifferential operator $P(D)$ of class $\{M_p\}$, $g \in \mathcal{E}^{\{N_p\}}(\Omega)$ and polynomial q ; this follows by induction on the degree of q . Assume for simplicity that $0 \in \Omega$. Write $H(x) = H(x_1, \dots, x_d) := H(x_1) \otimes \dots \otimes H(x_d)$, where $H(x_j)$ is the characteristic function of the positive half-axis, and $\text{p.v.}(x^{-1}) = \text{p.v.}(x_1^{-1}) \otimes \dots \otimes \text{p.v.}(x_d^{-1})$, where $\text{p.v.}(x_j^{-1})$ is the principle value regularization of the function x_j^{-1} . Let f be either $H(x)$ or $\text{p.v.}(x^{-1})$. By employing the global structural theorem for infrahyperfuctions of class $\{M_p\}$ [43], we find $g \in \mathcal{E}^{\{N_p\}}(\mathbb{R}^d)$ and an ultradifferential operator $P(D)$ of class

$\{M_p\}$ such that $P(D)g = f$. Set

$$\partial = \frac{\partial^d}{\partial x_1 \cdots \partial x_d} \quad \text{and} \quad q(x) = x_1 x_2 \cdots x_d.$$

The observation made at beginning of the proof now yields $q \circ \partial H = (q \partial H)$ and $q \circ \text{p.v.}(x^{-1}) = q \text{p.v.}(x^{-1})$. Since $q \partial H = 0$ and $q \text{p.v.}(x^{-1}) = 1$ in $\mathcal{B}^{\{M_p\}}(\Omega)$, we obtain

$$\partial H = \partial H \circ (q \circ \text{p.v.}(x^{-1})) = (\partial H \circ q) \circ \text{p.v.}(x^{-1}) = 0,$$

contradicting $\partial H = \delta \neq 0$ in $\mathcal{B}^{\{M_p\}}(\Omega)$ and the injectivity of $\mathcal{B}^{\{M_p\}}(\Omega) \rightarrow \mathcal{A}^{\{M_p\}, \{N_p\}}(\Omega)$. \square

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